JOURNAL OF COMBINATORIAL THEORY (B) **19**, 1–4 (1975)

Hamiltonian Circuits on the n-Dimensional Octahedron

DAVID SINGMASTER

Polytechnic of the South Bank, London, SE1 0AA, England

Communicated by F. Harary

Received June 26, 1974

An explicit formula and an asymptotic formula are obtained for the number of labeled Hamiltonian circuits on the n-dimensional octahedron (cross-polytope). Numerical results, bounds, and a conjectured asymptotic formula are given for the number of unlabeled circuits.

Hamiltonian circuits (HC's) on the n-cube have been studied by several authors. HC's on the n-octahedron do not seem to have been considered previously, which is surprising since the n-octahedron is combinatorially much easier than the n-cube. In addition, the n-octahedron is the dual of the n-cube, so that HC's on the faces of the n-cube are HC's on (the vertices of) the n-octahedron, yet neither viewpoint has been studied.

In this note, I present an explicit formula for $H(n)$, the number of labeled HC's on the n-octahedron, and I show that $H(n)$ is asymptotic to $(2n)!/e$. I will also state the number of unlabeled HC's for $2 \leqslant n \leqslant 8$ and will give a plausible asymptotic formula for this number.

For our purposes, we only consider the graph of the n-octahedron, which is the complete n-partite graph $K_{2,2,\ldots,2}$. This consists of n pairs of opposite points with edges connecting each point to every other point except its opposite. A labeled HC is a sequence of $2n$ vertices such that no opposite vertices are adjacent. Here we are considering the $2n$ vertices as distinct objects, i.e., they are labeled, and we consider the sequence as a cycle, i.e., the first and last positions are adjacent. Thus we can view our HC as a way of seating n couples around a circular table so that no man is next to his wife.

To count the number, $H(n)$, of such HC's, we apply inclusion and exclusion. First we consider all $(2n)!$ permutations of the vertices; then we exclude all ways of choosing a pair of adjacent positions, seating a couple in it, and seating the remaining people in any way. But arrangements with more than one adjacent couple will be multiply excluded, so we must include the ways of choosing two pairs of adjacent positions, seating two couples adjacently in them, and seating the remainder in any way, etc.

DEFINITION 1. Let $N(2n, k)$ be the number of ways of choosing k pairs of adjacent positions, choosing k couples and seating them adjacently in the positions, and seating the remaining people in any way.

Clearly $N(2n, 0) = (2n)!$ and our previous discussion gives

LEMMA 2. $H(n) = \sum_{k=0}^{n} (-1)^k N(2n, k)$.

PROPOSITION 3. $N(2n, k) = \binom{n}{k}[2n/(2n-k)]\, 2^k(2n-k)!$

Proof. We need the following result which arises in the "problème des ménages" [1, pp. 34–35]. The number of ways of selecting k positions, no two consecutive, from $2n$ positions arranged in a circle is $[2n/(2n-k)]\binom{2n-k}{k}$. Now we can select k couples from n in $\binom{n}{k}$ ways and these can be adjacently seated in the k selected positions and their immediate successors in $k!\, 2^k$ ways. The remaining $2n - 2k$ persons can be arranged in the remaining seats in $(2n - 2k)!$ ways. Multiplying all these factors together and simplifying gives the proposition.

THEOREM 4. *Thus we have*

$$H(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} [2n/(2n-k)]\, 2^k(2n-k)!$$

Values of $H(n)$ for $n = 2(1)8$ are given in Table I. Since the symmetry group of the n-octahedron has $n!\, 2^n$ elements, we can divide $H(n)$ by this factor, and I have given the ratio in the Table I.

TABLE I

n	$H(n)$	$H(n)/n!2^n$	$H(n)/(2n)!$	$G(n)$	$G(n)/g(n)$
2	8	1	.333333	1	8
3	192	4	.266666	2	6
4	11,904	31	.295238	7	3.614
5	1,125,120	293	.310053	29	1.980
6	153,262,080	3326	.319962	196	1.414
7	28,507,207,680	44,189	.326999	1788	1.133
8	6,951,513,784,320	673,471	.332246	21,994	1.045

At first I thought that $H(n)$ would be small in comparison with $(2n)!$ However, it is not difficult to show that $H(n) \geqslant 2n(2n-3)\,H(n-1)$, hence $H(n) \geqslant 2n(2n-2)! > (2n-1)!$ Upon computing the ratios $H(n)/(2n)!$ as also given in Table I, I conjectured that $H(n)/(2n)! \to \frac{1}{3}$. Allen Schwenk pointed out that $N(2n, k)/(2n)!$ is close to $1/k!$ so that we should expect $H(n)/(2n)! \to 1/e$. The following is the formalization of his remark.

THEOREM 5. $H(n)/(2n)! \to 1/e$.

Proof. A little simplification yields

$$\frac{N(2n, k)}{(2n)!} = \frac{n(n-1)\cdots(n-k-1)}{(2n-1)(2n-2)\cdots(2n-k)}\,\frac{2^k}{k!}.$$

For a fixed k, we have $\lim_{n\to\infty} N(2n, k)/(2n)! = 1/k!$ and we also easily see that $N(2n, k)/(2n)! \leqslant 1/k!$ for $k \geqslant 3$, for any n. This can be interpreted as a uniformity condition which justifies the interchange of $\lim_{n\to\infty}$ with $\sum_{k=0}$. I have not been able to locate an explicit statement of this, so I give the details.

Choose M so that $\sum_{k=M+1}^{\infty} 1/k! < \epsilon/3$. We have

$$\left|\frac{H(n)}{(2n)!} - \frac{1}{e}\right| \leqslant \sum_{k=M+1}^{\infty} \frac{N(2n, k)}{(2n)!} + \sum_{k=0}^{M}\left|\frac{1}{k!} - \frac{N(2n, k)}{(2n)!}\right| + \sum_{k=M+1}^{\infty}\frac{1}{k!}.$$

The first and last terms are $<\epsilon/3$ for all n. Since $\lim_{n\to\infty} N(2n, k)/(2n)! = 1/k!$ for each k and since our middle term involves only a finite number of k's, we can also make it $<\epsilon/3$ for all large enough n, which completes the proof.

The ratio $H(n)/(2n)!$ does not converge to $1/e$ very rapidly, in contrast to the problem of derangements. We have $1/e = .367879$, while $H(10)/20! = .339537$; $H(20)/40! = .353889$. For $n \geqslant 30$, some of the terms $N(2n, k)/(2n)!$ became too small for the computer I was using, so there may be some error in the following further values: $H(50)/100! = .362329$; $H(100)/200! = .365112$; $H(1000)/2000! = .367604$. One can apply Stirling's approximation to the theorem, but the result does not seem illuminating.

I have determined, by hand, the number, $G(n)$, of unlabeled HC's on the n-octahedron for $2 \leqslant n \leqslant 5$ and I have written a computer program which found $G(n)$ for $2 \leqslant n \leqslant 8$. These values are given in Table I.

Now two labeled HC's are equivalent as unlabeled HC's iff they are equivalent under the actions of the symmetries of the n-octahedron or under the actions of cyclically permuting or reversing a circuit. Because of the regularity of the n-octahedron, the symmetries partition the $H(n)$ labeled HC's into $H(n)/n!\,2^n$ classes with $n!\,2^n$ equivalent HC's. Then there are only $2 \cdot 2n$ remaining actions, so one of the above classes can be at most equivalent to $4n$ classes under the remaining actions. Thus we have shown

THEOREM 6. $H(n)/n!\,2^n \geqslant G(n) \geqslant H(n)/n!\,2^n 4n$.

(The identical relation holds on the n-cube when we replace the factor $4n$ by 2^{n+1}.) Let $H(n)/n!\,2^n 4n$ be denoted by $g(n)$. The ratios $G(n)/g(n)$ given in Table I indicate that $G(n)/g(n) \to 1$.

CONJECTURE 7. $G(n) \sim H(n)/n!\,2^n 4n$.

This conjecture is plausible both from the numerical evidence and because we know that it is unusual for HC's to be carried into themselves by cyclic permutation of the circuit or reversal of the circuit, so that we expect these $4n$ actions to generally carry the above classes into $4n$ classes.

ACKNOWLEDGMENT

Computing time was provided by the London Polytechnics Computer Unit.

REFERENCE

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